

A STEP TOWARDS THE ALEKSEEVSKII CONJECTURE

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ABSTRACT. We provide a reduction in the classification problem for non-compact, homogeneous, Einstein manifolds. Using this work, we verify the (Generalized) Alekseevskii Conjecture for a large class of homogeneous spaces.

A longstanding open question in the study of Riemannian homogeneous spaces is the classification of non-compact Einstein spaces. In the 1970s, it was conjectured by D. Alekseevskii that any (non-compact) homogeneous Einstein space of negative scalar curvature is diffeomorphic to \mathbb{R}^n . Equivalently, this conjecture can be phrased as follows:

Classical Alekseevskii Conjecture: Given a homogeneous Einstein space G/K with negative scalar curvature, K must be a maximal compact subgroup of G .

Part of the motivation for this conjecture comes from the statement holding true in the special cases of simply-connected Ricci flat homogeneous spaces [AK75], non-compact symmetric spaces, and, more generally, homogeneous (Einstein) spaces of negative sectional curvature [Ale75, AW76]. It is notable that in all these cases the manifolds are so-called solvmanifolds, i.e. they admit a transitive solvable group of isometries.

Since the posing of this conjecture, an enormous effort has been put into the classification of non-compact, homogeneous Einstein spaces. Among simply-connected solvable Lie groups with left-invariant metrics, much is known about existence and uniqueness of Einstein metrics, see [Heb98, Lau10] and references therein. Further, it follows from [AC99] that in the case of Einstein solvmanifolds with negative scalar curvature, one can reduce to the simply-connected case; see [Jab13b] for more in this direction.

In contrast to the progress made on solvmanifolds, there is work that suggests the conjecture might not be true. In [LdM82] it is shown that $SL(n, \mathbb{R})$ admits metrics of negative Ricci curvature for $n \geq 3$; although, in those examples the eigenvalues of Ric appear to be so widely spread that Einstein metrics might not exist. The case of transitive, unimodular groups of isometries was further investigated in [DM88] where it was shown that for such a space with negative Ricci curvature there must exist a transitive semi-simple group of isometries. For more in this direction, see Corollary 1.4 below.

The conjecture is known to hold in dimensions 4 and 5 and, in fact, all such spaces in these low dimensions are solvmanifolds, see [Jen69, Nik05]. In dimension 6, very recently it was shown that the conjecture holds in the presence of a transitive, non-unimodular group of isometries, see [AL13] or Section 4 of this work. Again, all such 6-dimensional spaces turn out to be solvmanifolds.

In the general setting, very little was known about the structure of homogeneous Einstein metrics until the recent work [LL12]. The work presented here builds on the structure results obtained there.

Refining the conjecture. We refine the question of which G and K are possible when G/K admits a G -invariant Einstein metric of negative scalar curvature. More precisely, assume G is

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simply-connected and consider a Levi decomposition

$$G = G_1 \ltimes G_2$$

where G_1 is semi-simple and G_2 is the (solvable) radical. Further, decompose G_1 into compact and non-compact factors, i.e.

$$G_1 = G_c G_{nc}$$

where G_c is the product of compact, simple, normal subgroups and G_{nc} the product of the non-compact, simple, normal subgroups. The intersection $G_c \cap G_{nc}$ is trivial since we assumed G to be simply-connected. Let K_{nc} be a choice of connected subgroup whose Lie algebra is a maximal compact subalgebra of \mathfrak{g}_{nc} (i.e. $Ad(K_{nc})$ is a maximal compact subgroup of $Ad(G_{nc})$). The subgroup K_{nc} is closed, although not necessarily compact.

Conjecture. *Let G/K be endowed with a G -invariant Einstein metric with negative scalar curvature. Then $G_c < K$ and (up to conjugation) $K_{nc} < K$.*

Observe that this is a stronger conjecture than the Alekseevskii Conjecture as the above implies that such an Einstein manifold has a transitive solvable group of isometries. We note that all known examples of homogeneous Einstein spaces with negative scalar curvature are isometric to solvable Lie groups with left-invariant metrics and it is believed by some that such spaces exhaust the class of non-compact, homogeneous, Einstein spaces. Further, the above conjecture holds for any transitive group of isometries of an Einstein solvmanifold, see [Jab13b]. We take a step towards resolving the above conjecture.

Theorem 0.1. *Let G/K be a homogeneous space endowed with a G -invariant Einstein metric of negative scalar curvature, then $G_c < K$.*

This result is established by applying the recent structural results of [LL12]. Building on that work and the above theorem, we have the following reduction in the classification problem.

Theorem 0.2. *Let G/K be a homogeneous Einstein space of negative scalar curvature. The transitive group G may be replaced with one with the following properties*

- (i) $G_1 = G_{nc}$ has no compact, normal subgroups,
- (ii) $K < G_1$, and
- (iii) *The radical decomposes as $G_2 = AN$, where the nilradical N (with the induced left-invariant metric) is nilsoliton, A is an abelian group, and $\text{ad } \mathfrak{a}$ acts by symmetric endomorphisms relative to the nilsoliton metric on \mathfrak{n} .*

We note that in the special case that \mathfrak{g} is solvable, (iii) above was already proven by Lauret in [Lau11].

Remark 0.3. *Applying [Jab13a] together with either of [HPW13] or [LL12], the above two theorems can be seen to apply more generally to homogeneous Ricci solitons on G/K .*

As an application of Theorem 0.2, we obtain a short proof of the recent result of Arroyo-Lafuente which verifies the Generalized Alekseevskii Conjecture in low dimensions, see Section 4.

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1. THE MAXIMUM PRINCIPLE

We apply the following well-known lemma through out.

Lemma 1.1. *Let X be a Killing field on a Riemannian manifold M and consider the function $f = \frac{1}{2}|X|^2$. Then*

$$\Delta f = |\nabla X|^2 - \text{ric}(X).$$

Furthermore, let $f(p)$ be a maximum of f and assume that X_p is tangent to a subspace of $T_p M$ along which the Ricci tensor is negative. Then by the Maximum Principle we must have $X = 0$.

We apply this lemma in the special case that $M = G/K$. Consider $\theta : \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g}_2)$ defined by $\theta(X) = \text{ad}(X)|_{\mathfrak{g}_2}$ and $\text{Ker } \theta = \{X \in \mathfrak{g} \mid \text{ad}(X)|_{\mathfrak{g}_2} = 0\}$. As $\mathfrak{g} = \mathfrak{g}_1 \ltimes \mathfrak{g}_2$ is a Levi decomposition of \mathfrak{g} , we see that $\theta(\mathfrak{g}_1) \ltimes \theta(\mathfrak{g}_2)$ is a Levi decomposition of $\theta(\mathfrak{g})$ and so $\theta(\mathfrak{g}_1) \cap \theta(\mathfrak{g}_2)$ is trivial. This yields

$$\text{Ker } \theta = (\text{Ker } \theta \cap \mathfrak{g}_1) + (\text{Ker } \theta \cap \mathfrak{g}_2).$$

The subalgebra $\text{Ker } \theta \cap \mathfrak{g}_1$ is an ideal of \mathfrak{g}_1 ; as such, $\text{Ker } \theta \cap \mathfrak{g}_1 = (\text{Ker } \theta \cap \mathfrak{g}_c) + (\text{Ker } \theta \cap \mathfrak{g}_{nc})$. Our interest is in the connected (normal) subgroup C of G with Lie algebra

$$\text{Lie } C = (\text{Ker } \theta \cap \mathfrak{g}_c) + \mathfrak{z}(\mathfrak{g}),$$

where $\mathfrak{z}(\mathfrak{g})$ denotes the center of \mathfrak{g} . Observe that G may be described as a product

$$G = C\overline{C},$$

where \overline{C} is a subgroup of G which commutes with C and such that $C \cap \overline{C} = Z(G)$. To see this, one builds \overline{C} from G_2 and the normal subgroups of G_1 which do not appear in C .

Lemma 1.2. *Let G/K be endowed with a metric of negative Ricci curvature in the directions tangent to $CK = C \cdot eK \subset G/K$, then $C < K$. Further, if G acts almost effectively, then C is the trivial group.*

Before proving this lemma, we state some corollaries that have not appeared in the literature before.

Corollary 1.3. *Let G/K be a homogeneous space where G acts almost effectively. In directions tangent to the orbit of $Z(G)$, the center of G , the Ricci curvature is non-negative. Furthermore, if G/K is endowed with a metric of negative Ricci curvature, then $Z(G)$ is discrete.*

In the special case of left-invariant metrics on Lie groups, this result is well-known [Mil76]. We do not know of the general homogeneous case appearing in the literature, although one can deduce an alternate proof to the corollary above using the techniques in [DM88]. Jorge Lauret has shown us a different proof of the corollary above which uses the relationship between Ric and the moment map.

Corollary 1.4. *Let G be a semi-simple group and assume that G/K is a homogeneous space of negative Ricci curvature. If G acts almost effectively on G/K , then G has no compact, normal subgroups.*

Consider G as in the corollary above and assume that G acts effectively on G/K , then the connected isometry group of G/K is simply G ; this follows from [Gor80] and generalizes [DM88, Corollary 3]. This gives some hope that the Alekseevskii Conjecture is indeed true as one would expect an Einstein space to have more symmetries than other metrics. Naturally, we apply the above in the special case of Einstein metrics.

Question 1.5. *Which semi-simple Lie groups admit left-invariant Einstein metrics?*

In the compact case, it is well-known that all such groups admit Einstein metrics, often more than one [Jen71]. In the non-compact case, one can quickly see by brute force that $SL_2\mathbb{R}$ does not admit a left-invariant Einstein metric. Until now, this was the only non-compact group for which the existence question had been answered.

Corollary 1.6. *Let G be a non-compact, semi-simple Lie group. If G has a compact, normal subgroup, then G does not admit a left-invariant Einstein metric.*

We now prove Lemma 1.2.

Proof. The proof of this lemma follows quickly from the preceding lemma. To apply that result, we take $X \in \text{Lie } C$ and consider the Killing field $[X]$ generated by X ; i.e.

$$[X]_p = \left. \frac{d}{dt} \right|_{t=0} \exp(tX) \cdot p$$

As $p = gK$ for some $g \in G$, and $g = c\bar{c}$ for $c \in C$ and $\bar{c} \in \overline{C}$, we see that

$$|[X]_{gK}| = |Ad(g^{-1})X| = |Ad(c^{-1})X|$$

Using that the group C is Ad -compact, we see that the function $f = \frac{1}{2}|[X]|^2$ does achieve a maximum. Further, by replacing X with $Ad(c^{-1})X \in \text{Lie } C$, we may assume that this maximum occurs at the point eK .

By hypothesis, the Ricci curvature is negative along the orbit $C \cdot eK$ and $\text{ric}(Ad(c^{-1})X) < 0$, unless $[X] = 0$. Applying Lemma 1.1, we see that $[X] = 0$. Thus $C < K$.

In the case that G acts effectively, $[X] = 0$ implies $X = 0$, and thus C must be trivial as it is connected. \square

We apply the results above in the special case of Einstein metrics to obtain our main results. First, we introduce our next main tool, the moment map.

2. THE MOMENT MAP

Let G be a real reductive Lie group acting linearly on a real vector space V . Denoting the G -action on V by ρ , we assume that V is endowed with an inner product $\langle \cdot, \cdot \rangle$ with the property

$$\rho(g)^t \in \rho(G) \quad \text{for all } g \in G,$$

where \cdot^t denotes the transpose relative to $\langle \cdot, \cdot \rangle$. Such inner products always exist for semi-simple G and, more generally, whenever $\rho(G) \subset \text{Aut}(V)$ is algebraically closed, see [Mos55]. We say such G is ‘self-adjoint’ relative to $\langle \cdot, \cdot \rangle$.

We may endow $\mathfrak{g} = \text{Lie } G$ with an inner product $\langle \langle \cdot, \cdot \rangle \rangle$ such that $Ad(G)$ is self-adjoint relative to $\langle \langle \cdot, \cdot \rangle \rangle$. Using these choices of inner products, and inspired by moment maps from symplectic and complex geometry [Nes84], we define the moment $m : V \rightarrow \mathfrak{g}$ of the G action on V by

$$\langle \langle m(v), X \rangle \rangle = \langle \rho(X)v, v \rangle,$$

for all $X \in \mathfrak{g}$ and $v \in V$. We note that we have abused notation and written the induced Lie algebra representation of $\mathfrak{g} = \text{Lie } G$ by the same symbol. Also, our definition of the moment map given above for a real reductive Lie group is fairly standard, although some authors would differ in that m would take values in the dual space \mathfrak{g}^* .

Remark 2.1. *In the sequel, we suppress ρ and denote $\rho(g)v$ by $g \cdot v$.*

Definition 2.2. *A point where the moment map vanishes is called a minimal point.*

Theorem 2.3. [RS90, Thm 4.3] *Let G be a reductive group acting linearly on V , as above, and p be a minimal point. The orbit $G \cdot p$ is closed and the stabilizer subgroup G_p is self-adjoint, i.e. closed under transpose. Furthermore, if $q \in G \cdot p$ is another minimal point, then $q \in K \cdot p$, where $K = G \cap O(V)$ and $O(V)$ is the orthogonal group relative to the inner product on V .*

We note that in the work of Richardson and Slodowy [RS90], they do not define the moment map. However, their definition of minimal point coincides with ours. Our interest in the tools above comes from looking at the change of basis action on the space of representations of a Lie algebra.

2.1. The space of representations. Given a Lie algebra \mathfrak{n} and vector space \mathfrak{h} , we consider the vector space $V = \text{Hom}(\mathfrak{h}, \text{Der}(\mathfrak{n}))$. If \mathfrak{h} were a Lie algebra acting by derivations on \mathfrak{n} , we could think of our representation of \mathfrak{h} as an element of the vector space $V = \text{Hom}(\mathfrak{h}, \text{Der}(\mathfrak{n}))$. In the sequel, \mathfrak{h} will not be a Lie algebra itself, but almost.

The group $GL(\mathfrak{n})$ acts in the natural way on V . For $\theta \in V$ and $g \in GL(\mathfrak{n})$ we define $g \cdot \theta$ by

$$(g \cdot \theta)(Y) = g\theta(Y)g^{-1},$$

for $Y \in \mathfrak{h}$. To define a moment map, we consider the most natural inner products on V and $\mathfrak{gl}(\mathfrak{n})$.

On $\mathfrak{gl}(\mathfrak{n})$, we use the usual inner product $\langle\langle A, B \rangle\rangle = \text{tr } AB^t$. To define an inner product on V , we first assume that \mathfrak{n} and \mathfrak{h} are endowed with inner products. Now take $\theta, \lambda \in V$ and $\{Y_i\}$ an orthonormal basis of \mathfrak{h} and define

$$\langle\theta, \lambda\rangle = \sum_i \text{tr } \theta(Y_i)\lambda(Y_i)^t,$$

where the transpose is being taken with respect to the inner product on \mathfrak{n} . To be able to define a moment map, we must first observe that indeed $\mathfrak{gl}(\mathfrak{n})$ is self-adjoint.

Remark 2.4. *If we denote this representation of $GL(\mathfrak{n})$ on V by ρ , then we have*

$$\rho(g^t) = \rho(g)^t,$$

where the first transpose is taken in $GL(\mathfrak{n})$ relative to the inner product on \mathfrak{n} and the second is taken in $GL(V)$ relative to our natural choice of inner product on $V = \text{Hom}(\mathfrak{h}, \text{Der}(\mathfrak{n}))$. In this way, the action of $GL(\mathfrak{n})$ on V is self-adjoint.

One quickly sees that the moment map $m : V \rightarrow \mathfrak{gl}(\mathfrak{n})$ of the $GL(\mathfrak{n})$ action on V is given by

$$m(\theta) = \sum_i [\theta(Y_i), \theta(Y_i)^t]$$

This is also computed in the appendix of [AL13] written by Jorge Lauret.

3. EINSTEIN METRICS ON G/K

Our main structure results on non-compact, homogeneous Einstein spaces build on those recently obtained in [LL12]. We begin by recalling the details from that work which we will use.

Let G/K be endowed with an Einstein metric of negative scalar curvature. The Lie algebra $\mathfrak{g} = \text{Lie } G$ admits a decomposition which is akin to an ‘algebraic Levi decomposition’. Namely, $\mathfrak{g} = \mathfrak{u} \ltimes \mathfrak{n}$, where \mathfrak{u} is reductive and \mathfrak{n} is the nilradical. In terms of the above notation, we have

$$\mathfrak{g}_1 = [\mathfrak{u}, \mathfrak{u}] \quad \text{and} \quad \mathfrak{g}_2 = \mathfrak{z}(\mathfrak{u}) \ltimes \mathfrak{n},$$

where $\mathfrak{z}(\mathfrak{u})$ denotes the center of \mathfrak{u} and \mathfrak{n} is the nilradical of \mathfrak{g} . Furthermore, $\mathfrak{k} \subset \mathfrak{u}$ and there exists an $Ad K$ -stable complement \mathfrak{h} of \mathfrak{k} in \mathfrak{u} . Naturally, we identify $\mathfrak{h} \oplus \mathfrak{n}$ with $T_{eK}G/K$ and we have that $\mathfrak{h} \perp \mathfrak{n}$.

Remark 3.1. The inner product on \mathfrak{h} extends naturally to an inner product on \mathfrak{u} such that $\mathfrak{h} \perp \mathfrak{k}$ and $Ad(K)$ acts orthogonally. As \mathfrak{u} is reductive, we have $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] + \mathfrak{z}(\mathfrak{u})$ (vector space direct sum). At this point in time, it is unknown if this direct sum is orthogonal.

The following comes from [LL12, Theorem 4.6]. In the sequel, G is a simply-connected Lie group, K is connected, and so G/K is simply-connected.

Theorem 3.2 (Lafuente-Lauret). *Let G/K be a homogeneous space endowed with an Einstein metric of negative scalar curvature. Then $G = U \ltimes N$, where U is reductive, N is nilpotent, and $K < U$. Further more,*

- (i) *The induced left-invariant metric on N is nilsoliton*
- (ii) *The induced metric on U/K satisfies $Ric_{U/K} = cId + C_\theta$, where C_θ is an operator which measures the action $\theta : U \rightarrow Aut(N)$ and is defined by*

$$\langle C_\theta(Y, Y) \rangle = \frac{1}{4} tr(S(\theta(Y))^2).$$

Here $S(A)$ denotes the symmetric part of the endomorphism $A : \mathfrak{n} \rightarrow \mathfrak{n}$ relative to the soliton metric on N .

- (iii) *The adjoint action of \mathfrak{u} on \mathfrak{n} , denoted by θ , satisfies the compatibility condition*

$$\sum_i [\theta(Y_i), \theta(Y_i)^t] = 0$$

where $\{Y_i\}$ is an orthonormal basis of $\mathfrak{h} \subset \mathfrak{u}$ and transpose is being taken with respect to the nilsoliton metric on \mathfrak{n} .

We may restrict θ to be a map $\theta : \mathfrak{h} \rightarrow Der(\mathfrak{n})$ and see that the compatibility condition above says precisely that θ is a minimal point of the $GL(\mathfrak{n})$ action on $V = Hom(\mathfrak{h}, Der(\mathfrak{n}))$.

Remark 3.3. Observe that by extending the inner product on $\mathfrak{h} \oplus \mathfrak{n}$ to an inner product on $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{n}$, as described above, we have that $ad \mathfrak{k}$ acts skew-symmetrically and so the point $\theta : \mathfrak{u} \rightarrow Der(\mathfrak{n})$ is a minimal point of the $GL(\mathfrak{n})$ action on $V = Hom(\mathfrak{u}, Der(\mathfrak{n}))$. We adopt this view in the sequel.

Taking an orthonormal basis of $\mathfrak{z}(\mathfrak{u})^\perp$ in \mathfrak{u} , we obtain a new inner product on the semi-simple algebra $[\mathfrak{u}, \mathfrak{u}]$. Namely, let $\{Y_i\}$ be an orthonormal basis of $\mathfrak{z}(\mathfrak{u})^\perp$ and write

$$Y_i = A_i + Z_i$$

where $A_i \in [\mathfrak{u}, \mathfrak{u}]$ and $Z_i \in \mathfrak{z}(\mathfrak{u})$. We define a new ‘associated’ inner product on $[\mathfrak{u}, \mathfrak{u}]$ by taking $\{A_i\}$ to be an orthonormal basis.

Lemma 3.4. *The compatibility condition (iii) of Theorem 3.2 is equivalent to*

- (i) *$\theta(\mathfrak{z}(\mathfrak{u}))$ consists of normal operators such that $\theta(\mathfrak{z}(\mathfrak{u}))^t$ commutes with all of $\theta(\mathfrak{u})$*
- (ii) *$\theta|_{[\mathfrak{u}, \mathfrak{u}]}$ is a minimal point of the $GL(\mathfrak{n})$ action on $Hom([\mathfrak{u}, \mathfrak{u}], Der(\mathfrak{n}))$, relative to the associated inner product on $[\mathfrak{u}, \mathfrak{u}]$.*

Proof. The first part of the lemma is an application of [RS90, Thm 4.3] (Theorem 2.3 above). To apply that work, we compute the stabilizer of the $GL(\mathfrak{n})$ -action at θ which is

$$GL(\mathfrak{n})_\theta = \{g \in GL(\mathfrak{n}) \mid g \cdot \theta = \theta\} = \{g \in GL(\mathfrak{n}) \mid g\theta(Y)g^{-1} = \theta(Y) \text{ for all } Y \in \mathfrak{u}\},$$

that is, $g \in GL(\mathfrak{n})_\theta$ must commute with all $\theta(Y)$. Now take $u \in Z(U)$ and consider $g = \theta(u)$. Clearly $\theta(u) \in (GL(\mathfrak{n}))_\theta$ and so, by the theorem above, we have the same for $(\theta(u))^t$. Upon differentiating, we obtain the first part of the lemma.

To see the second part of that lemma, consider a basis $\{Y_i\}$ of \mathfrak{u} which comes from one of $\mathfrak{z}(\mathfrak{u})$ and $\mathfrak{z}(\mathfrak{u})^\perp$. Applying the first part of the lemma, we see that

$$0 = \sum [\theta(Y_i), \theta(Y_i)^t] = \sum [\theta(A_i), \theta(A_i)^t],$$

which proves the lemma. □

Remark 3.5. *In the sequel, we will abuse notation and simply denote $\theta|_{[\mathfrak{u}, \mathfrak{u}]}$ by θ .*

As we see below, from the proof of Lemma 3.4 (i) comes part (iii) of Theorem 0.2.

Proposition 3.6. *Let G/K be a non-compact, homogeneous Einstein space. We may replace G with a transitive group such that $G_2 = AN$ and \mathfrak{a} acts on the nilradical \mathfrak{n} by symmetric endomorphisms (relative to the nilsoliton metric on \mathfrak{n}).*

Proof. Consider $\theta(u)^t$, for $u \in Z(U)$, and $\phi(u) = \theta(u)(\theta(u)^t)^{-1} \in \text{Aut}(N)$. From the above, we know that $\phi(u)$ acts orthogonally and commutes with $\theta(U)$. Thus we may realize $\phi(u) \in \text{Aut}(G)$ and consider

$$\overline{G} = \phi(Z(U)) \ltimes G$$

which acts naturally on G/K with stabilizer $\overline{K} = \phi(Z(U))K$. Take A to be the connected subgroup of \overline{G} whose Lie algebra is given by

$$\mathfrak{a} = \{X - \frac{1}{2}\phi(X) \mid X \in \mathfrak{z}(\mathfrak{u})\}.$$

By construction, the adjoint action of \mathfrak{a} on \mathfrak{n} is by symmetric operators. Observing that

$$\mathfrak{a} + \phi(\mathfrak{z}(\mathfrak{u})) = \mathfrak{z}(\mathfrak{u}) + \phi(\mathfrak{z}(\mathfrak{u})),$$

we have $[\mathfrak{u}, \mathfrak{u}] + \mathfrak{a} + \mathfrak{n} + \overline{\mathfrak{k}} = [\mathfrak{u}, \mathfrak{u}] + \mathfrak{z}(\mathfrak{u}) + \mathfrak{n} + \overline{\mathfrak{k}} = \overline{\mathfrak{g}}$ and so the group

$$[U, U]A \ltimes N$$

acts on $G/K = \overline{G}/\overline{K}$ with an open orbit. As this orbit will be a Riemannian homogeneous space, it is an open complete submanifold of the complete Riemannian manifold G/K and hence equals G/K . The group $[U, U]A \ltimes N$ is precisely the group we wished to construct. □

Proposition 3.7. *The stabilizer K is a subgroup of $G_1 = [U, U]$*

We leave the proof of this fact to the diligent reader, noting that it follows quickly upon using Corollary 1.3 to show that $\text{Ker } \theta \subset [\mathfrak{u}, \mathfrak{u}]$. This establishes part (ii) of Theorem 0.2 and the proof of Theorem 0.2 will be complete upon proving Theorem 0.1. Before proving that result, we need a series of technical lemmas.

3.1. Skew-symmetry of the \mathfrak{g}_c -action on \mathfrak{n} .

Lemma 3.8. *Let θ be a minimal point of the $GL(\mathfrak{n})$ -action on $V = \text{Hom}([\mathfrak{u}, \mathfrak{u}], \text{Der}(\mathfrak{n}))$, as in Lemma 3.4 (ii). Then $\theta(\mathfrak{g}_c)$ acts skew-symmetrically on $\mathfrak{n} = \text{nilrad}(\mathfrak{g}_2)$.*

Before proving this lemma, we establish some preliminary results.

Lemma 3.9. *There exists $g \in GL(\mathfrak{n})$ such that $(g \cdot \theta)([\mathfrak{u}, \mathfrak{u}])$ is self-adjoint.*

The above is equivalent to saying that there exists some inner product relative to which $\theta([\mathfrak{u}, \mathfrak{u}])$ is self-adjoint. However, the Lie algebra $\theta([\mathfrak{u}, \mathfrak{u}])$ is semi-simple and hence fully reducible and algebraic. For such Lie algebras, this is a classical result of Mostow [Mos55].

Let $\theta_1 = g \cdot \theta$, for g as in the lemma above. As $\theta_1([\mathfrak{u}, \mathfrak{u}])$ is self-adjoint, $\theta_1([\mathfrak{u}, \mathfrak{u}]) \cap \mathfrak{so}(\mathfrak{n})$ is a maximal compact subalgebra. Recall that the maximal compact subalgebras of $\theta_1([\mathfrak{u}, \mathfrak{u}])$ are all conjugate. As $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{g}_c + \mathfrak{g}_{nc}$, we see that $\theta_1(\mathfrak{g}_c)$ is contained in every maximal compact subalgebra and so we have the following.

Lemma 3.10. *The subalgebra $\mathfrak{g}_c^1 = \theta_1(\mathfrak{g}_c)$ of $\mathfrak{gl}(\mathfrak{n})$ consists of skew-symmetric endomorphisms.*

Notation: Given a group A with subgroup B , we denote the centralizer of B in A by $Z_A(B)$. Similarly, we define $Z_A(\mathfrak{b})$, where $\mathfrak{b} = \text{Lie } B$.

Lemma 3.11. *The group $H = Z_{GL(\mathfrak{n})}(\mathfrak{g}_c^1)$ is self-adjoint. Furthermore, the negative gradient flow of $\|m\|^2$ starting at θ_1 is tangent to the orbit $H \cdot \theta_1$, where m denotes the moment map of the $GL(\mathfrak{n})$ action on $V = \text{Hom}([\mathfrak{u}, \mathfrak{u}], \text{Der}(\mathfrak{n}))$.*

Proof. We prove the first claim. Take $h \in H$ and $X \in \mathfrak{g}_c^1$. Then we have

$$0 = [h, X]^t = [X^t, h^t] = -[X, h^t],$$

which proves the first statement. We continue with the second statement.

At a point $p \in V$, the gradient of $\|m\|^2$ is $4m(p) \cdot p$, where \cdot denotes the Lie algebra action of $\mathfrak{gl}(\mathfrak{n})$ on V . This well-known fact can be found in [Mar01, Lemma 2], where real algebraic groups are considered. Thus, our second claim follows from standard ODE theory arguments upon showing that

$$m(h \cdot \theta_1) \in \mathfrak{z}_{\mathfrak{gl}(\mathfrak{n})}(\mathfrak{g}_c^1) = \text{Lie } H \quad \text{for all } h \in H$$

Let $\{Y_i\}$ be an orthonormal basis of $[\mathfrak{u}, \mathfrak{u}]$ and write $Y_i = A_i + B_i$, where $A_i \in \mathfrak{g}_c$ and $B_i \in \mathfrak{g}_{nc}$. Then

$$\begin{aligned} m(h \cdot \theta_1) &= \sum [(h \cdot \theta_1)(Y_i), (h \cdot \theta_1)(Y_i)^t] \\ &= \sum [\theta_1(A_i) + h\theta_1(B_i)h^{-1}, (\theta_1(A_i) + h\theta_1(B_i)h^{-1})^t] \\ &= \sum [\theta_1(A_i) + h\theta_1(B_i)h^{-1}, -\theta_1(A_i) + (h\theta_1(B_i)h^{-1})^t] \\ &= \sum [h\theta_1(B_i)h^{-1}, (h\theta_1(B_i)h^{-1})^t] \end{aligned}$$

The second equality comes from the definition of H , namely that h commutes with $\mathfrak{g}_c^1 = \theta_1(\mathfrak{g}_c)$. The third inequality comes from Lemma 3.10. To see the fourth inequality, observe that $\theta_1(B_i) \in \mathfrak{h}$ since $[\mathfrak{g}_c, \mathfrak{g}_{nc}] = \{0\}$, H and \mathfrak{h} are closed under transpose, and \mathfrak{h} is a Lie algebra. Finally, $m(h \cdot \theta_1) \in \mathfrak{h}$ as \mathfrak{h} is a Lie algebra. \square

Lemma 3.12. *There exists $h \in H$ such that $h \cdot \theta_1$ is a minimal point.*

Proof. The orbit $GL(\mathfrak{n}) \cdot \theta_1 = GL(\mathfrak{n}) \cdot \theta$ is closed ([RS90, Theorem 4.3]) and so the negative gradient flow of $\|m\|^2$ (starting in the orbit) will converge to a point in the orbit which is a minimal point, see either [Jab12, Theorem 5.2] or [HSS08, Section 7]. (Note, these works are stated over projective space, but one easily passes back from the result in projective space to the vector space V .) As this flow is contained in the orbit $H \cdot \theta_1$, if we can show that $H \cdot \theta_1$ is closed, then we will be done. This is what we prove below. We warn the unfamiliar reader that it is not always the case that the orbit of a subgroup is also closed.

Our goal is to apply the Hilbert-Mumford criterion and we use the characterization of closed orbits given in [Jab11]; see the discussion between Proposition 6.6 and Theorem 6.8 of that work. Even though the work there applies in the general setting, it is described in a very special case and so we provide some details here.

To ease notational burden, we write Stab_{θ_1} for the stabilizer subgroup $GL(\mathfrak{n})_{\theta_1}$ of $GL(\mathfrak{n})$ at the point θ_1 . As $GL(\mathfrak{n}) \cdot \theta_1$ is closed, we see that Stab_{θ_1} is reductive (cf. [RS90, Theorem 4.3]). Now it is well-known that when Stab_{θ_1} is reductive, $GL(\mathfrak{n}) \cdot \theta_1$ is closed if and only if $Z_{GL(\mathfrak{n})}(\text{Stab}_{\theta_1}) \cdot \theta_1$ is closed. (For a proof of this fact, one can follow the aforementioned discussion in [Jab11] or see [Lun75] in the case of complex groups and then apply [Bir71, BHC62] to obtain the statement for real groups.)

The stabilizer subgroup of $Z_{GL(\mathfrak{n})}(Stab_{\theta_1})$ is $Z_{GL(\mathfrak{n})}(Stab_{\theta_1}) \cap Stab_{\theta_1} = Z(Stab_{\theta_1})$, the center of $Stab_{\theta_1}$. We define a group $I^{GL(\mathfrak{n})}$ such that $Z_{GL(\mathfrak{n})}(Stab_{\theta_1}) = I^{GL(\mathfrak{n})}Z(Stab_{\theta_1})$. This is done by taking the Lie group whose Lie algebra is defined by

$$\mathfrak{i}^{GL(\mathfrak{n})} = \{X \in \mathfrak{z}_{\mathfrak{gl}(\mathfrak{n})}(\mathfrak{stab}_{\theta_1}) \mid \text{tr}(XY) = 0 \text{ for all } Y \in \mathfrak{z}(\mathfrak{stab}_{\theta_1})\}$$

The group $I^{GL(\mathfrak{n})}$ is algebraic, the stabilizer subgroup is finite, and the orbit $I^{GL(\mathfrak{n})} \cdot \theta_1 = Z_{GL(\mathfrak{n})}(Stab_{\theta_1}) \cdot \theta_1$ is closed. As the stabilizer is finite, the Hilbert-Mumford criterion holds:

The orbit $I^{GL(\mathfrak{n})} \cdot \theta_1$ is closed if and only if $\lambda(t) \cdot \theta_1$ is closed for all algebraic 1-parameter subgroups $\lambda(t)$ of $I^{GL(\mathfrak{n})}$.

This criterion is well-known for complex algebraic groups and was proven for real algebraic groups by Birkes [Bir71].

We will now make an analogous set of statements for the group $H = Z_{GL(\mathfrak{n})}(\mathfrak{g}_c^1)$. As H is self-adjoint, it is reductive; H is also algebraic, being the centralizer of an algebraic group. Furthermore, the stabilizer of H coincides with the stabilizer of $GL(\mathfrak{n})$ at θ_1 since $Stab_{\theta_1} \subset H$. As this stabilizer is reductive as well, we have

$$H \cdot \theta_1 \text{ is closed if and only if } Z_H(Stab_{\theta_1}) \cdot \theta_1 \text{ is closed.}$$

Observe that the stabilizer of $Z_H(Stab_{\theta_1})$ at θ_1 is $Z(Stab_{\theta_1})$. Now we construct an algebraic group I^H whose Lie algebra is defined by

$$\mathfrak{i}^H = \{X \in \mathfrak{z}_{\mathfrak{h}}(\mathfrak{stab}_{\theta_1}) \mid \text{tr}(XY) = 0 \text{ for all } Y \in \mathfrak{z}(\mathfrak{stab}_{\theta_1})\}$$

As before, the stabilizer subgroup of I^H is finite and $I^H \cdot \theta_1 = Z_H(Stab_{\theta_1}) \cdot \theta_1$. Again, we have the Hilbert-Mumford criterion:

The orbit $I^H \cdot \theta_1$ is closed if and only if $\lambda(t) \cdot \theta_1$ is closed for all algebraic 1-parameter subgroups $\lambda(t)$ of I^H .

From the definitions of I^H and $I^{GL(\mathfrak{n})}$, we see that

$$I^H \subset I^{GL(\mathfrak{n})}.$$

In general, this does not happen with subgroups of $GL(\mathfrak{n})$; it happens in our case precisely because $Stab_{\theta_1} \subset H$. Passing through the two Hilbert-Mumford criteria, we see that $I^{GL(\mathfrak{n})} \cdot \theta_1$ being closed implies $I^H \cdot \theta_1$ is closed. But then this implies $H \cdot \theta_1$ is closed, as was to be shown. \square

Using Lemma 3.12, we now prove Lemma 3.8. As $h \cdot \theta_1 = (hg) \cdot \theta$ and θ are minimal points in the same $GL(\mathfrak{n})$ orbit, we know from [RS90, Theorem 4.3] that there exists $k \in O(\mathfrak{n})$ such that

$$(hg) \cdot \theta = k \cdot \theta.$$

Observe that $\theta(Y)$ is skew-symmetric if and only if $k\theta(Y)k^{-1}$ is so. But for $Y \in \mathfrak{g}_c$, we have

$$k\theta(Y)k^{-1} = hg\theta(Y)g^{-1}h^{-1} = g\theta(Y)g^{-1},$$

where the last equality follows from the definition of H , namely that it commutes with $\mathfrak{g}_c^1 = \theta_1(\mathfrak{g}_c)$. Applying Lemma 3.10, we see that the skew-symmetry of $g\theta(Y)g^{-1}$ gives that $\theta(\mathfrak{g}_c)$ is skew-symmetric.

3.2. Proof of main result. Using Lemma 3.8 and work of Lafuente-Lauret, see Theorem 3.2, we finish the proof of Theorem 0.1. Let G/K be endowed with an Einstein metric of negative scalar curvature and consider the induced geometry on U/K . From Theorem 3.2, we have

$$Ric_{U/K} = cId + C_\theta$$

where $\langle C_\theta(Y, Y) \rangle = \frac{1}{4}tr(S(\theta(Y)))^2$. However, Lemma 3.8 shows that

$$Ric_{U/K}(Y) < 0 \quad \text{for non-zero } Y \in \mathfrak{g}_c$$

Applying Lemma 1.2, we see that this is not possible and so $G_c < K$, as desired. This proves Theorem 0.1.

4. THE GENERALIZED ALEKSEEVSKII CONJECTURE IN DIMENSION 5

We apply the work above to show that any 5-dimensional, homogeneous Ricci soliton with negative cosmological constant is isometric to a simply-connected solvable Lie group with left-invariant metric. This verifies the Generalized Alekseevskii Conjecture in dimension 5. This result was recently obtained in [AL13] and we give an alternative proof.

We begin by restricting our attention to those Ricci solitons which are not Einstein as the Einstein case was previously established in [Nik05]. Further, we are able to restrict ourselves to the case that G/K is simply-connected as we will show that the spaces of interest are solvmanifolds. See [Jab13b] for this reduction to the simply-connected case.

From [Jab13a] together with either [HPW13] or [LL12], we know that there exists $G' > G$ such that G is codimension 1 in G' and G'/K is Einstein with non-trivial mean curvature vector H which satisfies

$$\langle H, X \rangle = tr(ad X) \quad \text{for all } X \in \mathfrak{g}.$$

Remark 4.1. *To obtain the desired result on 5-dimensional Ricci solitons, we study 6-dimensional, homogeneous Einstein spaces of negative scalar curvature with non-trivial mean curvature vector as G'/K being a solvmanifold implies the same is true for G/K . In the sequel G/K will denote such a 6-dimensional space.*

From Theorem 0.2 we have the decomposition

$$\mathfrak{g} = (\mathfrak{g}_1 + \mathfrak{z}(\mathfrak{u})) \ltimes \mathfrak{n}$$

where $\mathfrak{u} = \mathfrak{g}_1 + \mathfrak{z}(\mathfrak{u})$ is reductive, \mathfrak{g}_1 is semi-simple with no compact ideals, and $\mathfrak{k} \subset \mathfrak{g}_1$. Further, we have that the mean curvature vector is central in \mathfrak{u} , i.e. $H \in \mathfrak{z}(\mathfrak{u})$ (see Eqn. 2.1 of [Jab13a]).

Remark 4.2. *To show that G/K is a solvmanifold, it suffices to show that G_1/K is a solvmanifold.*

The nilradical \mathfrak{n} cannot be trivial as otherwise we would have an Einstein metric on U/K when U has non-trivial center (cf. Corollary 1.3). Now we see that $\dim G_1/K \leq 4$.

Case: $\dim \mathfrak{n} = 1$. In this case, we have $Der(\mathfrak{n}) \simeq \mathbb{R}$ is spanned by the mean curvature vector H and so $\theta(\mathfrak{g}_1) = 0$. If $\dim G_1/K = 4$, then from Lafuente-Lauret's structure theorem (Theorem 3.2) we see that G_1/K is Einstein and so our result is true as all 4-dimensional non-compact, homogeneous Einstein spaces are solvmanifolds. (See [Jen69] for the classification of 4-dimensional, non-compact, homogeneous Einstein spaces.)

Now assume that $\dim G_1/K = 3$. In this case, as $\dim \mathfrak{z}(\mathfrak{u}) = 2$, we consider the codimension 1 subgroup \overline{G} of G with Lie algebra

$$\overline{\mathfrak{g}} = \mathfrak{g}_1 + \mathbb{R}(X), \quad \text{where } X \in \mathfrak{z}(\mathfrak{u}) \text{ and } X \perp H.$$

As $tr(ad X) = 0$ (from the definition of the mean curvature vector H), we see that $\theta(X) = 0$. Further, we have that

$$Ric_{\overline{G}/K} < 0$$

To see this, one applies [LL12, Lemma 4.2] together with Lafuente-Lauret's structure theorem (Theorem 3.2) and the observation that $G/K = \overline{G}/K \times \mathbb{R}$ is a Riemannian product, where \mathbb{R} is the Lie group whose algebra contains the mean curvature vector H . However, the group \overline{G} has center and this violates Corollary 1.3.

Case: $\dim \mathfrak{n} = 2$. Here $\dim G_1/K = 3$ and the only case in which G_1/K is not already a solvmanifold is when $\mathfrak{g}_1 = \mathfrak{sl}_2\mathbb{R}$ with \mathfrak{k} trivial. We demonstrate that this case is not a possibility.

From [LL12, Theorem 4.6] we know that $\theta(\mathfrak{sl}(2, \mathbb{R}))$ belongs to a self-adjoint subalgebra of $\text{Der}(\mathfrak{n})$. However, as $\mathfrak{n} = \mathbb{R}^2$, we see that $\text{Der}(\mathfrak{n}) = \mathfrak{gl}(2, \mathbb{R})$ and so $\theta(\mathfrak{sl}(2, \mathbb{R}))$ must be self-adjoint itself. As such, there exists $X \in \mathfrak{sl}(2, \mathbb{R})$ such that $\theta(X)$ is skew-symmetric and so the smallest eigenvalue of

$$\text{Ric}_{\widetilde{SL(2, \mathbb{R})}}(v, v) = \text{Ric}_{U/K}(v, v) = c|v|^2 + \text{tr}(S(\theta(v)))^2$$

occurs in the direction of X which is tangent to a maximal compact subalgebra of $\mathfrak{sl}(2, \mathbb{R})$. Note that $\text{ad } X$ has purely imaginary eigenvalues.

One can see that no such metric exists on $\widetilde{SL(2, \mathbb{R})}$ by applying [Mil76]. From that work, we know that for any left-invariant metric on $SL(2, \mathbb{R})$ there exists an orthonormal basis $\{e_1, e_2, e_3\}$ such that

$$[e_2, e_3] = \lambda_1 e_1 \quad [e_3, e_1] = \lambda_2 e_2 \quad [e_1, e_2] = \lambda_3 e_3$$

with $\lambda_1 < 0 < \lambda_2 \leq \lambda_3$. Furthermore, the given basis diagonalizes $\text{Ric}_{\widetilde{SL(2, \mathbb{R})}}$.

For any left-invariant metric on $\widetilde{SL(2, \mathbb{R})}$, we claim that the smallest eigenvalue of $\text{Ric}_{\widetilde{SL(2, \mathbb{R})}}$ occurs in either the e_2 or e_3 direction. This follows quickly from [Mil76] by using that either two of the eigenvalues of $\text{Ric}_{\widetilde{SL(2, \mathbb{R})}}$ are zero and one negative, or two are negative and one positive. In the first case, one can show that the eigenvalues are $\text{ric}(e_1) = \text{ric}(e_3) = 0$ and $\text{ric}(e_2) = 2\lambda_3\lambda_1 < 0$. In the case that all eigenvalues of $\text{Ric}_{\widetilde{SL(2, \mathbb{R})}}$ are non-zero, these eigenvalues are given by $\text{ric}(e_i) = 2\mu_{i+1}\mu_{i+2}$ with $\mu_i = \frac{1}{2}(-\lambda_i + \lambda_{i+1} + \lambda_{i+2})$. (Here the formulas are written using the convention that our indices are taken mod 3.) By inspection, one is able to see that $\text{ric}(e_1)$ cannot be the smallest such eigenvalue.

Finally, $\text{ad } e_2$ and $\text{ad } e_3$ have real eigenvalues. Together with the above work we see that $\text{ad } X$ above must have only zero eigenvalues and so X is central in $\mathfrak{sl}(2, \mathbb{R})$. This is a contradiction as semi-simple Lie algebras have no center.

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